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On Exponential Splines

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1. INTRODUCTION AND DESCRIPTION OF *B*-SPLINES

Among the various classes of splines, the polynomial spline has been received the greatest attention primarily because it admits a basis of *B*-splines which can be accurately and efficiently computed. Recently it has been shown that trigonometric and hyperbolic splines also admit *B*-spline bases ([1, 2]).

The object of the present paper is to give *B*-spline bases for hyperbolic and trigonometric splines which are little different from the hyperbolic and trigonometric ones in [1, 2]. Throughout this paper, we assume that m is a natural number and λ is a positive parameter. First we consider a sequence $\{d_{m,j}\}$ defined by

$$\begin{aligned} &d_{m,j} = d_{m-1,j} + d_{m-1,j-1}, \qquad d_{m,0} = d_{m,m} = 1 \qquad (m \geqslant 3) \\ &d_{m,j} = 0 \qquad (j \leqslant -1 \text{ and } j \geqslant m+1) \\ &d_{2,0} = 1, \qquad d_{2,1} = 2\cosh\lambda, \qquad d_{2,2} = 1. \end{aligned} \tag{1.1}$$

By a simple calculation, we have

$$\lim_{\lambda \to 0} d_{m,j} = \binom{m}{j}. \tag{1.2}$$

Now, by making use of the constant $d_{m,j}$, we may define a hyperbolic *B*-spline $Q_{m+1,\lambda}$ of degree m:

for m odd:

$$\begin{aligned} Q_{m+1,\lambda}(x) = \sum_{j=0}^{m+1} (-1)^j d_{m+1,j} (1/\lambda^m) \Big[&\sinh \lambda(x-j)_+ \\ &- \lambda(x-j)_+ - \cdots - \frac{\{\lambda(x-j)_+\}^{m-2}}{(m-2)!} \Big]; \end{aligned} \tag{1.3}$$

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for m even:

$$Q_{m+1,\lambda}(x)=\sum_{j=0}^{m+1}(-1)^j d_{m+1,j}(1/\lambda^m)\left[\cosh\lambda(x-j)_+ -1-\cdots-\frac{\{\lambda(x-j)_+\}^{m-2}}{(m-2)!}\right]. \tag{1.4}$$

Here we shall prove that the hyperbolic B-spline $Q_{m+1,\lambda}$ is characterized by a convolution process of an exponential function ϕ_λ and a characteristic function χ on $[0, 1)$, where

$$\begin{aligned}\phi_\lambda(x)&=e^{\lambda x} \quad (0\leqslant x<1) \quad \text{and} \quad 0 \quad (\text{otherwise})\\ \chi(x)&=1 \quad (0\leqslant x<1) \quad \text{and} \quad 0 \quad (\text{otherwise}).\end{aligned} \tag{1.5}$$

THEOREM 1.

$$Q_{m+1,\lambda}(x)=(\underbrace{\chi*\chi*\cdots*\chi}_{m-1}*\phi_\lambda*\phi_{-\lambda})(x) \tag{1.6}$$

*where * means the convolution of two functions, i.e.,*

$$(f*g)(x)=\int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt.$$

Proof. By the definition of $d_{m,j}$, we have

$$Q'_{m+1,\lambda}(x)=Q_{m,\lambda}(x)-Q_{m,\lambda}(x-1). \tag{1.7}$$

Since $Q_{m+1,\lambda}(0)=0$, from above we obtain

$$\begin{aligned}Q_{m+1,\lambda}(x)&=\int_{x-1}^{x} Q_{m,\lambda}(t)\, dt=\int_0^1 \chi(t)\, Q_{m,\lambda}(x-t)\, dt\\ &=(\chi*Q_{m,\lambda})(x)=\cdots=(\underbrace{\chi*\chi*\cdots*\chi}_{m-1}*Q_{2,\lambda})(x).\end{aligned} \tag{1.8}$$

By a simple calculation, we have

$$\begin{aligned}\lambda Q_{2,\lambda}(x)&=\sinh\lambda x_+ -\cosh\lambda\sinh\lambda(x-1)_+ +\sinh\lambda(x-2)_+\\ &=\begin{cases}\sinh\lambda x & (0\leqslant x\leqslant 1)\\ \sinh\lambda(2-x) & (1\leqslant x\leqslant 2)\end{cases}\\ &=\lambda(\phi_\lambda*\phi_{-\lambda})(x).\end{aligned} \tag{1.9}$$

This completes the proof of Theorem 1.

Next we shall give a trigonometric B-spline $\tilde{Q}_{m+1,\lambda}$ by replacing the parameter λ in the definition of the hyperbolic B-spline $Q_{m+1,\lambda}$ by $i\lambda$ ($i = \sqrt{-1}$):

for m odd:

$$\tilde{Q}_{m+1,\lambda}(x) = (-1)^{(1/2)(m-1)} \sum_{j=0}^{m+1} (-1)^j \tilde{d}_{m+1,j}(1/\lambda^m) \Bigg[\sin \lambda(x-j)_+ - \lambda(x-j)_+ - \cdots - (-1)^{(1/2)(m+1)} \frac{\{\lambda(x-j)_+\}^{m-2}}{(m-2)!} \Bigg]; \tag{1.10}$$

for m even:

$$\tilde{Q}_{m+1,\lambda}(x) = (-1)^{(1/2)m} \sum_{j=0}^{m+1} (-1)^j \tilde{d}_{m+1,j}(1/\lambda^m) \Bigg[\cos \lambda(x-j)_+ - 1 - \cdots - (-1)^{(1/2)(m+2)} \frac{\{\lambda(x-j)_+\}^{m-2}}{(m-2)!} \Bigg] \tag{1.11}$$

where $\{\tilde{d}_{m,j}\}$ is defined by the same recursion formula (1.1) with $2\cos\lambda$ in the definition of $\tilde{d}_{2,2}$.

Similarly as in the hyperbolic B-spline, we have:

THEOREM 2.

$$\tilde{Q}_{m+1,\lambda}(x) = (\underbrace{\chi * \chi * \cdots * \chi}_{m-1} * Q_{i\lambda} * \phi_{-i\lambda})(x), \tag{1.12}$$

where

$$\begin{aligned} (\phi_{i\lambda} * \phi_{-i\lambda})(x) &= \sin \lambda x/\lambda \qquad (0 \leqslant x \leqslant 1) \\ &= \sin \lambda(2-x)/\lambda \qquad (1 \leqslant x \leqslant 2). \end{aligned} \tag{1.13}$$

By making use of Theorems 1 and 2, we may easily have the properties of these hyperbolic and trigonometric B-splines similar to those of the polynomial ones ([3]). In addition, we have the following theorems that imply $1, x, ..., x^{m-2}$, $\cosh \lambda x$ and $\sinh \lambda x \in \operatorname{Span}\{Q_{m+1,\lambda}(x-j)\}_{j=-\infty}^{\infty}$.

THEOREM 3. *For* $m \geqslant 2$,

$$1, x, ..., x^{m-2} \in \operatorname{Span}\{Q_{m+1,\lambda}(x-j)\}_{j=-\infty}^{\infty}. \tag{1.14}$$

Proof. By the definition of the characteristic function, we have

$$\sum_{j=-\infty}^{\infty} \chi(x-j)=1. \tag{1.15}$$

A successive convolution of this partition of unity and $\chi,\ldots,\chi$, ϕ_λ and $\phi_{-\lambda}$ yields

$$\begin{aligned}\sum_{j=-\infty}^{\infty} Q_{m+1,\lambda}(x-j) &= (1*\underbrace{\chi*\chi*\cdots*\chi}_{m-2}*\phi_\lambda*\phi_{-\lambda})(x)\\ &=\{\sinh(\tfrac{1}{2}\lambda)/(\tfrac{1}{2}\lambda)\}^2 \qquad (m\geqslant 2).\end{aligned} \tag{1.16}$$

In addition, since $j-(j-1)=1$, from above we have

$$\begin{aligned}&\{\sinh(\tfrac{1}{2}\lambda)/(\tfrac{1}{2}\lambda)\}^2\\ &\quad=\sum_{j=-\infty}^{\infty} jQ_{m+1,\lambda}(x-j)-\sum_{j=-\infty}^{\infty}(j-1)\,Q_{m+1,\lambda}(x-j)\\ &\quad=\sum_{j=-\infty}^{\infty} j\{Q_{m+1,\lambda}(x-j)-Q_{m+1,\lambda}(x-j-1)\}\\ &\quad=\sum_{j=-\infty}^{\infty} jQ'_{m+2,\lambda}(x-j).\end{aligned} \tag{1.17}$$

Integration of the above equation from 0 to x gives

$$\begin{aligned}&\sum_{j=-\infty}^{\infty} jQ_{m+2,\lambda}(x-j)-\sum_{j=-\infty}^{\infty} jQ_{m+2,\lambda}(-j)\\ &\qquad=\{\sinh(\tfrac{1}{2}\lambda)/(\tfrac{1}{2}\lambda))\}^2 x\end{aligned} \tag{1.18}$$

where

$$\begin{aligned}&\sum_{j=-\infty}^{\infty} Q_{m+2,\lambda}(-j)\\ &\quad=-\{Q_{m+2,\lambda}(1)+2Q_{m+2,\lambda}(2)+\cdots+(m+1)\,Q_{m+2,\lambda}(m+1)\}\\ &\quad=-(\tfrac{1}{2}m+1)\{\sinh(\tfrac{1}{2}\lambda)/(\tfrac{1}{2}\lambda)\}^2.\end{aligned} \tag{1.19}$$

During the above computation, we used

$$\left.\begin{aligned}Q_{m+2,\lambda}(j)&=Q_{m+2,\lambda}(m+2-j)\\ \sum_{j=-\infty}^{\infty} Q_{m+2,\lambda}(x-j)&=\{\sinh(\tfrac{1}{2}\lambda)/(\tfrac{1}{2}\lambda)\}^2.\end{aligned}\right\} \tag{1.20}$$

Thus we have for $m \geqslant 2$

$$\sum_{j=-\infty}^{\infty} (j+\tfrac{1}{2}m+1)\, Q_{m+2,\lambda}(x-j) = \{\sinh(\tfrac{1}{2}\lambda)/(\tfrac{1}{2}\lambda)\}^2 x \qquad (1.21)$$

i.e.,

$$x \in \operatorname{Span}\{Q_{m+1,\lambda}(x-j)\}_{j=-\infty}^{\infty} \qquad (\text{for } m \geqslant 3). \qquad (1.22)$$

By making use of a simple identity:

$$k \prod_{p=0}^{k-2} (j+p) = \prod_{p=0}^{k-1} (j+p) - \prod_{p=0}^{k-1} (j+p-1) \qquad (1.23)$$

i.e.,

$$2j = (j+1)\, j - j(j-1)$$
$$3(j+1)\, j = (j+2)(j+1)\, j - (j+1)\, j(j-1)$$
$$\cdots,$$

inductively we have the desired result.

THEOREM 4. *For* $m \geqslant 2$,

$$\begin{aligned}\sum_{j=-\infty}^{\infty} Q_{m,\lambda}(x-j) \cosh \lambda(j+\tfrac{1}{2}m) &= [\{(2/\lambda)\sinh \tfrac{1}{2}\lambda\}^{m-3}\cosh \tfrac{1}{2}\lambda] \cosh \lambda x \\ \sum_{j=-\infty}^{\infty} Q_{m,\lambda}(x-j) \sinh \lambda(j+\tfrac{1}{2}m) &= [\{(2/\lambda)\sinh \tfrac{1}{2}\lambda\}^{m-3}\cosh \tfrac{1}{2}\lambda] \sinh \lambda x.\end{aligned} \qquad (1.24)$$

Proof. For $m=2$, by an elementary computation we have the above relations. Multiplying the first relation by $2\sinh \tfrac{1}{2}\lambda$, we have for $m \geqslant 3$

$$\sum_{j=-\infty}^{\infty} Q'_{m+1,\lambda}(x-j) \sinh \lambda\{j+\tfrac{1}{2}(m+1)\} = \{(2/\lambda)\sinh \tfrac{1}{2}\lambda\}^{m-3}\, 2\sinh \tfrac{1}{2}\lambda \cos \tfrac{1}{2}\lambda \cosh \lambda x \qquad (1.25)$$

where

$$Q'_{m+1,\lambda}(x) = Q_{m,\lambda}(x) - Q_{m,\lambda}(x-1).$$

Integration of the above equation from 0 to x yields

$$\sum_{j=-\infty}^{\infty} Q_{m+1,\lambda}(x-j)\sinh\lambda\{j+\tfrac{1}{2}(m+1)\}+c$$
$$=\{(2/\lambda)\sinh\tfrac{1}{2}\lambda\}^{m-2}\cosh\tfrac{1}{2}\lambda\sinh\lambda x. \tag{1.26}$$

Here we shall prove the above constant c to be zero. By Theorem 1, we obtain

$$Q_{m+1,\lambda}(x)=\int_{x-2}^{x} Q_{m-1}(t)\,\psi(x-t)\,dt \tag{1.27}$$

where

$$Q_{m-1}(x)=(\underbrace{\chi*\chi*\cdots*\chi}_{m-1})(x),$$

i.e., Q_{m-1} is the polynomial B-spline of degree $m-2$, and $\psi(x)=(\phi_\lambda*\phi_{-\lambda})(x)$. From (1.27), by a simple calculation we get

$$(D^2-\lambda^2)\,Q_{m+1,\lambda}(x)=Q_{m-1,\lambda}(x)-2\cosh\lambda\,Q_{m-1,\lambda}(x-1)$$
$$+Q_{m-1,\lambda}(x-2)\qquad(m\geqslant 3). \tag{1.28}$$

Operating the differential operator $(D^2-\lambda^2)$ to the both sides of (1.26), we have

$$\sum_{j=-\infty}^{\infty}[\sinh\lambda\{(j+\tfrac{1}{2}(m+1)\}$$
$$-2\cosh\lambda\sinh\lambda\{(j-1+\tfrac{1}{2}(m+1)\}$$
$$+\sinh\lambda\{(j-2+\tfrac{1}{2}(m+1)\}]$$
$$\times Q_{m-1,\lambda}(x-j)-c\lambda^2=0. \tag{1.29}$$

Since the coefficient of $Q_{m-1,\lambda}(x-j)$ is identically zero, we have the desired second relation with m replaced by $m+1$. Similarly we have the first relation with $m+1$ from the second with m.
For $\tilde{Q}_{m+1,\lambda}(x)$, we have

$$\sum_{j=-\infty}^{\infty}\tilde{Q}_{m+1,\lambda}(x-j)=\{\sin(\tfrac{1}{2}\lambda)/(\tfrac{1}{2}\lambda)\}^2\qquad(m\geqslant 2) \tag{1.30}$$

from which follows

$$1\in\operatorname{Span}\{\tilde{Q}_{m+1,\lambda}(x-j)\}_{j=-\infty}^{\infty}\qquad(m\geqslant 2)$$
$$\text{for}\quad\lambda\neq 2k\pi\qquad(k=1,2,\ldots). \tag{1.31}$$

In addition, as in the proof of Theorem 3 we have

$$x, x^2, \ldots, x^{m-2} \in \mathrm{Span}\{\tilde{Q}_{m+1,\lambda}(x-j)\}_{j=-\infty}^{\infty}$$
$$\text{for } \lambda \neq 2k\pi \qquad (k=1, 2, \ldots). \tag{1.32}$$

For $\tilde{Q}_{m,\lambda}(x)$ $(m \geqslant 2)$, we have

$$\begin{aligned} &\sum_{j=-\infty}^{\infty} \tilde{Q}_{m,\lambda}(x-j)\cos\lambda(j+\tfrac{1}{2}m) \\ &\quad = [\{(2/\lambda)\sin\tfrac{1}{2}\lambda\}^{m-3}\cos\tfrac{1}{2}\lambda]\cos\lambda x \\ &\sum_{j=-\infty}^{\infty} \tilde{Q}_{m,\lambda}(x-j)\sin\lambda(j+\tfrac{1}{2}m) \\ &\quad = [\{(2/\lambda)\sin\tfrac{1}{2}\lambda\}^{m-3}\cos\tfrac{1}{2}\lambda]\sin\lambda x \end{aligned} \tag{1.33}$$

from which follows

$$\cos\lambda x \text{ and } \sin\lambda x \in \mathrm{Span}\{\tilde{Q}_{m,\lambda}(x-j)\}_{j=-\infty}^{\infty} \qquad (m \geqslant 2)$$
$$\text{for } \lambda \neq k\pi \qquad (k=1, 2, \ldots). \tag{1.34}$$

2. An Application of a Hyperbolic Spline of Degree 4 to a Numerical Solution of a Simple Perturbation Problem

We are concerned with numerical solution of a simple singular perturbation problem:

$$\varepsilon y''(x) - y(x) = g(x) \qquad (0 \leqslant x \leqslant 1) \tag{2.1}$$

$$y(0) = \alpha, \qquad y(1) = \beta \tag{2.2}$$

with $0 < \varepsilon \ll 1$.

Miller has proposed and proved the convergence, uniformly in ε, of the difference scheme:

$$\varepsilon\left\{\frac{\frac{1}{2}\lambda}{\sinh(\frac{1}{2}\lambda)}\right\}^2 \frac{(y_{j+1} - 2y_j + y_{j-1})}{h^2} - y_j = g_j$$
$$(1 \leqslant j \leqslant n-1) \tag{2.3}$$

$$y_0 = \alpha, \qquad y_n = \beta \qquad \text{with} \quad \lambda = h/\sqrt{\varepsilon} \tag{2.4}$$

where for a natural number n, $h = 1/n$ and $g_j = g(jh)$.

Now, by making use of the hyperbolic B-spline $Q_{5,\lambda}$ of degree 4 we consider a spline function of the form

$$s(x)=\sum_{j=-4}^{n-1} \alpha_j Q_{5,\lambda}(x/h-j), \qquad \lambda=h/\sqrt{\varepsilon} \tag{2.5}$$

with undetermined coefficients $(\alpha_{-4}, \alpha_{-3},..., \alpha_{n-1})$. The above $s(x)$ will be an approximate solution if it satisfies

$$\varepsilon s_j''-s_j=g_j \qquad (1\leqslant j\leqslant n-1) \tag{2.6}$$

$$s_0=\alpha, \qquad s_n=\beta. \tag{2.7}$$

In order to transform the above collocation method (2.6)–(2.7) to a difference method, we shall require the following consistency relation obtained by use of the similar technique for the polynomial spline:

$$\begin{aligned}(\#)\ h^{-2}&(a_{2,4}s_{j-1}+a_{2,3}s_j+a_{2,2}s_{j+1}+s_{2,1}s_{j+2})\\ &=(a_{0,4}s_{j-1}''+a_{0,3}s_j''+a_{0,2}s_{j+1}''+a_{0,1}s_{j+2}'')\end{aligned} \tag{2.8}$$

where $a_{k,j}=Q_{5,\lambda}^{(k)}(j)$ $(1\leqslant k\leqslant 3)$.

Since $Q_{5,\lambda}^{(k)}(j)=Q_{5,\lambda}^{(k)}(5-j)$ $(k=0, 2)$, the above relation $(\#)$ at consecutive four mesh points is reduced to a short term relation at consecutive three mesh points, by making an alternating sum of $(\#)$ obtained by writing down $(\#)$, subtracting $(\#)$ with j replaced by $j+1$, adding $(\#)$ with j replaced by $j+2$ and so on. That is,

$$\begin{aligned}h^{-2}&\mu(\lambda)(s_{j+1}-2s_j+s_{j-1})\\ &=s_{j+1}''+\{\mu(\lambda)-2\}\, s_j''+s_{j-1}''\end{aligned} \tag{2.9}$$

where

$$\mu(\lambda)=\frac{\lambda^2(\cosh\lambda-1)}{\cosh\lambda-1-\frac{1}{2}\lambda^2}>12. \tag{2.10}$$

By means of this short term consistency relation, the collocation method (2.6)–(2.7) is equivalent to a difference method:

$$\begin{aligned}\varepsilon\mu\,&\frac{(s_{j+1}-2s_j+s_{j-1})}{h^2}-\{s_{j+1}+(\mu-2)\,s_j+s_{j-1}\}\\ &=g_{j+1}+(\mu-2)\,g_j+g_{j-1} \qquad (1\leqslant j\leqslant n-1)\end{aligned} \tag{2.11}$$

$$s_0=\alpha, \qquad s_n=\beta \qquad \text{with} \quad \mu=\mu(h/\sqrt{\varepsilon}). \tag{2.12}$$

TABLE ($\varepsilon = 10^{-4}$)

Observed Errors at Mesh Points in Collocation Method (2.11)–(2.12) and Miller's Difference Method (2.3)–(2.4)

Method	Collocation		Difference	
$x \backslash h$	1/20	1/40	1/20	1/40
0.05	0.922-5*	0.955-6	−0.301-1	−0.840-2
0.1	0.790-5	0.818-6	−0.132-2	−0.181-3
0.2	0.302-5	0.313-6	−0.310-5	−0.120-5
0.3	−0.3.2-5	−0.313-6	−0.498-5	−0.125-5
0.4	−0.790-5	−0.818-6	−0.130-4	−0.367-5
0.5	−0.977-5	−0.101-5	−0.161-4	−0.404-5

* We denote 0.922×10^{-5} by 0.922-5, and the errors mean (exact values) − (approximate values).

The solution of (2.1)–(2.2) would be dominated by terms of $e^{\pm x/\sqrt{\varepsilon}}$, and so in order to derive a lumped mass system of the above difference scheme we let

$$(s_{j+1} + g_{j+1}) + (s_{j-1} + g_{j-1}) \cong (e^{h/\sqrt{\varepsilon}} + e^{-h/\sqrt{\varepsilon}}) \times (s_j + g_j) = 2\cosh(h/\sqrt{\varepsilon})(s_j + g_j). \quad (2.13)$$

Since

$$\mu(h/\sqrt{\varepsilon}) - 2 + 2\cosh(h/\sqrt{\varepsilon}) = \left\{\frac{\sinh(\frac{1}{2}\lambda)}{\frac{1}{2}\lambda}\right\}^2 \mu(h/\sqrt{\varepsilon}), \quad (2.14)$$

we have a lumped mass system of (2.6)–(2.7) which is identical with the above Miller's difference scheme (2.3)–(2.4).

Now we consider an application of the difference scheme (2.11)–(2.12) to a simple two point boundary value problem:

$$\varepsilon y'' - y = \cos^2 \pi x + 2\pi^2 \varepsilon \cos 2\pi x, \qquad y(0) = y(1) = 0. \quad (2.15)$$

The exact solution is given by

$$y(x) = \frac{\exp((x-1)/\sqrt{\varepsilon}) + \exp(-x/\sqrt{\varepsilon})}{1 + \exp(-1/\sqrt{\varepsilon})} - \cos^2 \pi x.$$

References

1. T. Lyche and R. Winther, A stable recurrence relation for trigonometric *B*-splines, *J. Approx. Theory* **25** (1979), 266–279.
2. L. L. Schumaker, On hyperbolic splines, *J. Approx. Theory* **38** (1983), 144–166.
3. L. L. Schumaker, "Spline Functions: Basic Theory," Wiley, New York, 1981.
4. J. J. Miller, On the convergence, uniformly in ε, of difference schemes for a two point boundary singular perturbation problem, *in* "Proceedings, Conf., Nijmegen," Academic Press, New York, 1979.
5. J. Stoer and R. Bulirsch, "Introduction to Numerical Analysis," Springer-Verlag, Berlin/New York/Heidelberg, 1979.